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STEENROD AND DYER-LASHOF OPERATIONS ON BU

BY

TIMOTHY LANCE[1]

ABSTRACT. This paper describes a simple, fast algorithm for the computation of Steenrod and Dyer-Lashof operations on BU. The calculations are carried out in $H^*(BU, \mathbf{Z}_{(p)})$ and $H_*(BU, \mathbf{Z}_{(p)})$ where p local lifts are determined by the values on primitives and Cartan formulas. This algorithm also provides a description of Steenrod and Dyer-Lashof operations on the fiber of any H map (or infinite loop map) $BU \to BU$, and applications to the classifying spaces of surgery which arise in this fashion will appear shortly.

1. Introduction. This paper describes a simple, fast algorithm for the computation of Steenrod and Dyer-Lashof operations on BU. The calculation is made by lifting to cohomology and homology with coefficients in $\mathbf{Z}_{(p)}$, the integers localized at p, where polynomial bases can be defined directly in terms of the primitives. These bases occur naturally in the study of the fibers of H maps $f\colon BU \to BU$, and applications of this work in the computation of the bordism and description of the geometry of the classifying spaces of surgery which arise in this fashion will appear shortly.

For convenience we assume throughout that p is an odd prime, although the same arguments work for $p = 2$ with minor modifications. Let $\{d_{e_1}, d_{e_2}, \ldots\}$ and $\{c_{e_1}, c_{e_2}, \ldots\}$ denote the usual bases for the primitives in $H_*(BU, \mathbf{Z}_{(p)})$ and $H^*(BU, \mathbf{Z}_{(p)})$, respectively. For example, d_{e_n} is dual in the basis of monomials to the nth Chern class c_n.

Let $T_k(t_0, t_1, \ldots, t_k) = t_0^{p^k} + pt_1^{p^{k-1}} + \cdots + p^k t_k$ be the kth Witt polynomial. Using the Waring formula for the primitives and some elementary number theory of multinomial coefficients we show that the equations

$$d_{e_n p^k} = T_k(a_{n,0}, a_{n,1}, \ldots, a_{n,k})$$

and

$$c_{e_n p^k} = T_k(a^*_{n,0}, a^*_{n,1}, \ldots, a^*_{n,k})$$

inductively define bases $\{a_{n,k}\}$ and $\{a^*_{n,k}\}$ for $H_*(BU, \mathbf{Z}_{(p)})$ and $H^*(BU, \mathbf{Z}_{(p)})$, respectively, where n ranges over the positive integers which are prime to p and $k \geq 0$. Our general approach is to use the above formulas to define maps of polynomial algebras with the correct values on primitives, and then check that the maps are indeed well defined for $\mathbf{Z}_{(p)}$ coefficients and have the desired mod p reductions. For the

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Steenrod operations we obtain the following (where $\binom{a}{b}$ equals the binomial coefficient if $0 \leq b \leq a$ and 0 otherwise).

THEOREM 1. *There is a map of polynomial algebras* $P = P^0 + P^1 + \cdots : H^*(BU, \mathbf{Z}_{(p)}) \to H^*(BU, \mathbf{Z}_{(p)})$ *defined inductively for n prime to p and* $k \geq 0$ *by*

$$\sum_{s=0}^{\infty} \binom{np^k}{s} c_{e_{np^k+s(p-1)}} = T_k(Pa^*_{n,0}, \ldots, Pa^*_{n,k}).$$

The component homomorphisms P^s: $H^m(BU, \mathbf{Z}_{(p)}) \to H^{m+2s(p-1)}(BU, \mathbf{Z}_{(p)})$ *reduce* mod p *to Steenrod operations. Dually, there is a map* $P_* = P^0_* + P^1_* + \cdots : H_*(BU, \mathbf{Z}_{(p)}) \to H_*(BU, \mathbf{Z}_{(p)})$ *of polynomial algebras defined inductively by*

$$\sum_{s=0}^{\infty} \binom{np^k - s(p-1)}{s} \frac{np^k}{np^k - s(p-1)} d_{e_{np^k-s(p-1)}} = T_k(P_*a_{n,0}, \ldots, P_*a_{n,k})$$

where the component maps P^s_*: $H_m(BU, \mathbf{Z}_{(p)}) \to H_{m-2s(p-1)}(BU, \mathbf{Z}_{(p)})$ *reduce* mod p *to the dual Steenrod operations.*

The proof is quite easy thanks to a result of Borel and Serre [3] giving $P^s(C_n)$ as an integral polynomial on the Chern classes defined by its effect on elementary symmetric functions.

THEOREM 2. *There is a ring homomorphism* $Q = Q^0 + Q^1 + \cdots : H_*(BU, \mathbf{Z}_{(p)}) \to H_{**}(BU, \mathbf{Z}_{(p)})$ *defined inductively for n prime to p and* $k \geq 0$ *by*

$$\sum_{r=0}^{\infty} (-1)^{n+r} \binom{r-1}{np^k - 1} d_{e_{np^k+r(p-1)}} = T_k(Qa_{n,0}, \ldots, Qa_{n,k}).$$

The component maps Q^n: $H_m(BU, \mathbf{Z}_{(p)}) \to H_{m+2r(p-1)}(BU, \mathbf{Z}_{(p)})$ *reduce modulo p to the Dyer-Lashof operations.*

Here $H_{**}(BU, \mathbf{Z}_{(p)})$ denotes the ring of formal series $X_0 + X_1 + X_2 + \cdots$ with $X_i \in H_{2i}(BU, \mathbf{Z}_{(p)})$. For the proof we must rely on Kochman's [8] computation of $Q^r(d_{e_m})$ to define our lift, and then check that its mod p reduction satisfies the conditions of his algorithm. The hardest verification is of the Nishida relations; the proof makes use of the particular lift P^s_* defined above and a monstrous identity involving binomial coefficients whose proof was supplied by Leonard Carlitz.

The construction of the lifted map Q requires only that it is a ring homomorphism with the right values on primitives. Hence by working p locally instead of mod p the list of necessary properties in Kochman's algorithm ([8, Theorem 97] or Theorem 4.1 in this paper) can be trimmed substantially.

COROLLARY. *There is an algorithm for computing the p-local lift* Q^r: $H_m(BU, \mathbf{Z}_{(p)}) \to H_{m+2r(p-1)}(BU, \mathbf{Z}_{(p)})$ *of the Dyer-Lashof operation using the following properties*:

(1) Q^r *is linear for all* $r \geq 0$.

(2) $Q^r(xy) = \sum_i (Q^i x)(Q^{r-i} y)$ *for all* $r \geq 0$.

(3) $Q^r(d_{e_m}) = \binom{r-1}{m-1} d_{e_{m+r(p-1)}}$ *for* $r \geq 0$, $m > 0$.

Several spaces of geometric interest, such as $\mathrm{Im}(J)$ or the factor N of $PL/0$ with

$\pi_k(N)$ equal to the p torsion in bP^{k+1}, are defined as the fibers of p local H maps $f: BU \to BU$. Such maps are studied in [9], and a simple description of the cohomology of the fiber F is given in terms of the basis elements $a^*_{n,k}$ and their suspensions. In [11] the algorithm of Theorem 1 is then applied in an Adams spectral sequence computation of the complex bordism of N and the spaces in its Postnikov tower. If f is an infinite loop map then Theorem 2 provides a similar description of the Dyer-Lashof operations on F. This is exploited in [10], particularly in the construction of smoothings classified by N.

A number of detailed calculations of these operations on BO and BU have already appeared. Brown, Peterson, and Davis have some partial descriptions of the Steenrod operations in [4, 5], while in [17] Peterson obtains a formula for $P^s(C_n)$ in terms of Chern classes and certain symmetric polynomials. Closed formulas for the Dyer-Lashof operations have been obtained by Priddy [18] for $p = 2$ and D. Moore [15] for $p > 2$ extending some low dimensional computations of Kochman [8]. Shay [20] constructs integral lifts of both families of operations, deriving closed expressions using the Newton and Waring formulae. The algorithms above also yield closed formulas in terms of the bases $\{a_{n,j}\}$ and $\{a^*_{n,j}\}$ which are relatively efficient; $Q^r(a_{n,k})$, for example, is a polynomial of $\leq (r - mp^k + 1)^{p^k}$ terms. All of these formulas, however, are complicated and difficult to work with. For the applications in [10, 11] it is essential to have the simple recursive description of P and Q above.

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2. Witt polynomials and the homology of classifying spaces. Given indeterminants $t_0, t_1, \ldots$ and a fixed odd prime p we define the kth Witt polynomial at p by

$$T_k(t) = t_0^{p^k} + pt_1^{p^{k-1}} + \cdots + p^k t_k$$

where we abbreviate $t = (t_0, t_1, \ldots)$. Classically, these polynomials were used to invert functors which one might not have suspected were invertible [2]. For us their usefulness stems from the following p-integrality result, where R denotes either the integers or a p-local ring and $t^p = (t_0^p, t_1^p, \ldots)$.

2.1. LEMMA. *Let $g_0, g_1, g_2, \ldots$ be polynomials or formal power series in $t_0, t_1, \ldots$ with coefficients in R such that $g_k(t) \equiv g_{k-1}(t^p) \bmod p^k$, $k = 1, 2, \ldots$. Then the equations*

$$g_k(t) = T_k(\varphi_0(t), \varphi_1(t), \ldots, \varphi_k(t))$$

inductively define polynomials or formal series with coefficients in R.

PROOF. Since $\varphi_0 = g_0$, we suppose inductively that $\varphi_0, \ldots, \varphi_{k-1}$ are polynomials in R for some $k > 0$. But $T_k(t) = T_{k-1}(t^p) + p^k t_k$, so we must verify that $g_k(t) - T_{k-1}((\varphi_0(t))^p, \ldots, (\varphi_{k-1}(t))^p)$ vanishes mod p^k.

Evidently the polynomials $(\varphi_j(t))^p$ and $\varphi_j(t^p)$ are congruent mod p. Applying the binomial theorem inductively it follows that $(\varphi_j(t))^{p^{j+1}}$ and $(\varphi_j(t^p))^{p^j}$ are congruent mod p^{j+1}. Consequently,

$$T_{k-1}((\varphi_0(t))^p, \ldots, (\varphi_{k-1}(t))^p) \equiv T_{k-1}(\varphi_0(t^p), \ldots, \varphi_{k-1}(t^p)) \mod p^k.$$

But $g_{k-1}(t) = T_{k-1}(\varphi_0(t), \ldots, \varphi_{k-1}(t))$ by definition, and so

$$g_{k-1}(t^p) = T_{k-1}(\varphi_0(t^p), \ldots, \varphi_{k-1}(t^p)).$$

The lemma now follows from the congruence $g_k(t) \equiv g_{k-1}(t^p) \mod p^k$. □

We use this to describe a p local decomposition of the homology and cohomology of BU. Recall that $H^*(BU, R)$ is a polynomial Hopf algebra $R[c_1, c_2, \ldots]$ on the universal Chern classes whose coproduct, coming from the Whitney sum map μ, is given by $\mu^* c_n = \sum c_i \otimes c_{n-i}$. If $d_n \in H_{2n}(BU, R)$ is dual (in the basis of monomials) to c_1^n, then the correspondence $c_n \to d_n$ defines an isomorphism of Hopf algebras $H^*(BU, R) \to H_*(BU, R)$.

For any n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ of weight $w(\alpha) = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ let $c^\alpha = c_1^{\alpha_1} \cdots c_n^{\alpha_n} \in H^{2w(\alpha)}(BU, \mathbf{Z}_{(p)})$ and $d^\alpha = d_1^{\alpha_1} \cdots d_n^{\alpha_n} \in H_{2w(\alpha)}(BU, \mathbf{Z}_{(p)})$ denote the cup and Pontrjagin products, respectively. Denote the classes dual (in the basis of monomials) to c^α and d^α by d_α and c_α, respectively. We describe these classes directly. Let part(α) denote the partition $I = i_1, \ldots, i_r$ of $w(\alpha)$ in which the number j appears exactly α_j times, and define S_I to be the unique polynomial satisfying $S_I(\sigma_1, \ldots, \sigma_{w(\alpha)}) = \sum t_1^{i_1} \cdots t_r^{i_r}$ where $\sigma_1, \sigma_2, \ldots$ are the elementary symmetric polynomials in $t_1, \ldots, t_m$, $m \geq w(\alpha)$ (we follow the notation of [**14**, p. 188]). Then by a straightforward generalization of arguments in [**12**] we obtain the following.

2.2. LEMMA. $c_\alpha = S_{\text{part}(\alpha)}(c_1, \ldots, c_{2w(\alpha)})$ *with coproduct*

$$\mu^* c_\alpha = \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1} \otimes c_{\alpha_2}.$$

The primitives of $H^*(BU, R)$ are generated as an R module by $c_{e_1}, c_{e_2}, \ldots$ where $e_n = (0, 0, \ldots, 0, 1)$ is the nth unit vector. These are given explicitly by the Waring formula

$$c_{e_n} = S_n(c_1, \ldots, c_n) = \sum_{w(\alpha)=n} (-1)^{|\alpha|+n} \frac{n}{|\alpha|} \{\alpha\} c^\alpha$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\{\alpha\} = |\alpha|!/(\alpha_1! \cdots \alpha_n!)$. The corresponding statements about d_α and d_{e_n} are also true.

2.3. THEOREM [**7, 9**]. *For any p local ring R and n prime to p the equations*

$$c_{e_n p^k} = T_k(a^*_{n,0}, \ldots, a^*_{n,k})$$

and

$$d_{e_n p^k} = T_k(a_{n,0}, \ldots, a_{n,k})$$

inductively define elements $a^*_{n,k} \in H^{2np^k}(BU, R)$ and $a_{n,k} \in H_{2np^k}(BU, R)$. If A_n^* denotes the polynomial Hopf algebra $R[a^*_{n,0}, a^*_{n,1}, \ldots]$ and $A_n = R[a_{n,0}, a_{n,1}, \ldots]$, then there

are canonical isomorphisms of Hopf algebras

$$H^*(BU, R) \cong \bigotimes_{n \text{ prime to } p} A_n^*$$

and

$$H_*(BU, R) \cong \bigotimes_{n \text{ prime to } p} A_n.$$

PROOF. We first check that $a_{n,k}$ is p-integral. By Lemmas 2.1 and 2.2 it suffices to show that

$$S_{np^k}(d_1, \ldots, d_{np^k}) - S_{np^{k-1}}(d_1^p, \ldots, d_{np^{k-1}}^p)$$

vanishes mod p^k. But for any α of weight np^k with some entry prime to p the coefficient of d^α is given by the Waring formula as $\pm np^k\{\alpha\}/|\alpha|$, and $\{\alpha\}/|\alpha|$ is p integral by Proposition 5.1. If $\alpha = p\beta$, then the coefficient of d^α is $\pm np^{k-1}(\{p\beta\} - \{\beta\})/|\beta|$ (apply Lemma 2.2 twice). But $(\{p\beta\} - \{\beta\})/|\beta|$ is p divisible in $\mathbf{Z}_{(p)}$ by Proposition 5.2, so $a_{n,k}$ is well defined. Since $a_{n,k} = nd_{np^k}$ + decomposables by the Waring formula again, for any p local R the elements $a_{n,k}$ with n prime to p and $k \geq 0$ form a polynomial basis of $H_*(BU, R)$.

Note that $A_n \otimes \mathbf{Q}$ is clearly a sub Hopf algebra of $H_*(BU, \mathbf{Q})$ since both are primitively generated, and hence A_n is a sub Hopf algebra of the torsion free $H_*(BU, \mathbf{Z}_{(p)})$. It follows that the map above is an isomorphism of Hopf algebras. The proof for cohomology is the same. □

The subalgebras A_n and A_n^* are in fact isomorphic bipolynomial Hopf algebras. Identify $H^*(BU, \mathbf{Z}_{(p)})$ with the dual of $H_*(BU, \mathbf{Z}_{(p)})$ and give the latter free $\mathbf{Z}_{(p)}$ module the basis of all monomials in the elements $a_{n,j}$ for n prime to p and $j \geq 0$. The dual of any element of A_n lies in A_n^*; for example, $((-1)^{n+1}/n)c_{e_{np^k}}$ is dual to $a_{n,k}$. The correspondence $a_{n,j} \to a_{n,j}^*$ defines an isomorphism of Hopf algebras.

3. Lifting the Steenrod operations. Let P^s: $H^q(X, \mathbf{Z}/p) \to H^{q+2s(p-1)}(X, \mathbf{Z}/p)$ denote the Steenrod reduced pth powers with dual operations

$$P_*^s\colon H_{q+2s(p-1)}(X, \mathbf{Z}/p) \to H_q(X, \mathbf{Z}/p)$$

(i.e. $P_*^s = \operatorname{Hom}(P^s, 1)$ where

$$H_*(X, \mathbf{Z}/p) = \operatorname{Hom}_{\mathbf{Z}/p}(H^*(X, \mathbf{Z}/p), \mathbf{Z}/p)).$$

For $X = BU$ Borel and Serre described the action of P^s as follows.

3.1. THEOREM [3]. $P^s c_n = S_{\text{part}(n-s, 0, \ldots, 0, s)}(c_1, \ldots, c_{n+s(p-1)})$ (*the s is in the pth position*).

This is usually treated as a mod p formula and in principle determines P^s on $H^*(BU, \mathbf{Z}/p)$ because of the Cartan formula. In what follows we regard Theorem 3.1 as an integral lift of the Steenrod operation on c_n and examine its p local properties. In particular, we assume all calculations are carried out in $H^*(BU, \mathbf{Z}_{(p)})$.

3.2. COROLLARY.

$$P^s c_n = \frac{n + s(p-1)}{n}\binom{n}{s} c_{n+s(p-1)} + \textit{decomposables.}$$

PROOF. The fundamental symmetry principle of symmetrical algebra [13, Volume 1] states that, given α, β of equal weight, the coefficient of c^α in $S_{\text{part}(\beta)}(c)$ equals the coefficient of c^β in $S_{\text{part}(\alpha)}(c)$. In particular, the coefficient of $c_{n+s(p-1)}$ in $P^s(c_n)$ equals that of $c_1^{n-s}c_p^s$ in $S_{n+s(p-1)}(c)$. By Lemma 2.2 this equals $[(n + s(p-1))/n]\binom{n}{s}$. □

Note that this result agrees modulo p with Brown and Peterson's $P^s c_n = \binom{n-1}{s} c_{n+s(p-1)}$ + decomposables [4] (and dually with Kochman's Lemma 96 in [8]). Their formula, however, cannot be satisfied integrally by any family of lifts

$$P^s\colon H^*(BU, \mathbf{Z}_{(p)}) \to H^*(BU, \mathbf{Z}_{(p)})$$

which send primitives to primitives and satisfy the Cartan formula. If such lifts do exist and satisfy Corollary 3.2 then $P^s(c_{e_n}) = \binom{n}{s} c_{e_{n+s(p-1)}}$ since $c_{e_n} = (-1)^{n+1} n c_n$ + decomposables by Lemma 2.2. We use this to construct the desired lift.

3.3. THEOREM. *There is a map of polynomial algebras* $P = P^0 + P^1 + \cdots$: $H^*(BU, \mathbf{Z}_{(p)}) \to H^*(BU, \mathbf{Z}_{(p)})$ *defined inductively for n prime to p and* $k \geq 0$ *by*

$$\sum_{s=0}^{\infty}\binom{np^k}{s} c_{e_{np^k+s(p-1)}} = T_k(Pa^*_{n,0}, \ldots, Pa^*_{n,k}).$$

The component homomorphisms

$$P^s\colon H^m(BU, \mathbf{Z}_{(p)}) \to H^{m+2s(p-1)}(BU, \mathbf{Z}_{(p)})$$

satisfy multiplicative and comultiplicative Cartan formulae, vanish if $2s > m$, *are given on the Chern classes by Theorem* 3.1, *and reduce* mod p *to the Steenrod operations.*

The comultiplicative Cartan formula (or co-Cartan formula) states that $P^s\mu^* x = \sum_i \sum (P^i x') \otimes (P^{s-i} x'')$ where $\mu^* x = \sum x' \otimes x''$ and follows modulo p by naturality and the Cartan formula. The Adem relations seem to be an intrinsically mod p result and do not lift to $\mathbf{Z}_{(p)}$ coefficients.

PROOF. We first check that P is well defined. By Theorem 2.3 we may write the degree j term of

$$\sum_s \binom{np^k}{s} c_{e_{np^k+s(p-1)}}$$

as a polynomial $\Phi_{k,j}(a)$ where a denotes the sequence, ordered by degree, of polynomial generators $a^*_{m,i}$ for m prime to p and $i \geq 0$. By Lemma 2.1 it suffices to show that $\Phi_{k,j}(a)$ vanishes mod p^k if j is prime to p, and $\Phi_{k,j}(a) \equiv \Phi_{k-1,j/p}(a^p)$ mod p^k when p divides j.

Since

$$\Phi_{k,j}(a) = \binom{np^k}{s} c_{e_{np^k+s(p-1)}}$$

with s prime to p when j is, the first requirement follows by Proposition 5.1. Thus suppose p^i is the highest power of p dividing j for some $i > 0$, and set $m = j/p^i$,

$j = np^k + s(p-1)$, and $a_m = (a_{m,0}, a_{m,1}, \ldots)$. Then

$$\Phi_{k,j}(a) - \Phi_{k-1,j/p}(a^p) = \binom{np^k}{s}\left(T_i(a_m) - T_{i-1}(a_m^p)\right) + \left(\binom{np^k}{s} - \binom{np^{k-1}}{s/p}\right) T_{i-1}(a_m^p).$$

But

$$\binom{np^k}{s} \equiv \binom{np^{k-1}}{s/p} \mod p^k$$

by Proposition 5.2, while $T_i(a) - T_{i-1}(a^p) = p^i a_{m,i}$. If $i \geq k$ we are done, and if $i < k$ then p^i exactly divides s (i.e. no higher power divides it) so that

$$\binom{np^k}{s} \equiv 0 \mod p^{k-i}$$

by Proposition 5.2.

Since P is a map of polynomial algebras, the component maps $P^s\colon H^m(BU, \mathbf{Z}_{(p)}) \to H^{m+2s(p-1)}(BU, \mathbf{Z}_{(p)})$ satisfy the Cartan formula. By convention $\binom{np^k}{s} = 0$ if $s > np^k$ so that P^s vanishes for $s > m$. To verify the co-Cartan formula and the value of P^s on c_n, let $\bar{P} = \bar{P}_0 + \bar{P}_1 + \cdots$ denote the map of polynomial algebras given on the Chern classes by Theorem 3.1. We show that $P = \bar{P}$ by checking that they agree on primitives. But by Theorem 3.1 and Lemma 2.2

$$\begin{aligned}\mu^*\bar{P}c_n &= \sum_s \mu^* c_{(n-s,0,\ldots,0,s)} \\ &= \sum_{s_1,s_2,i} c_{(i-s_1,0,\ldots,0,i)} \otimes c_{(n-i-s_2,0,\ldots,0,s_2)} \\ &= \sum_i \bar{P}c_i \otimes \bar{P}c_{n-i}.\end{aligned}$$

If follows that $\mu^*\bar{P} = (\bar{P} \otimes \bar{P})\mu^*$ (that is, $\bar{P}$ satisfies the co-Cartan formula) and hence that $\bar{P}$ sends primitives to primitives. Using Lemma 2.2 and Corollary 3.2 it follows that $\bar{P}^s c_{e_n} = \binom{n}{s} c_{e_{n+s(p-1)}} = P^s c_{e_n}$, and hence $P = \bar{P}$. □

By dualizing we obtain a lift of the Steenrod homology operations P^s_*. This particular lift will be crucial in determining that the maps constructed in the next section actually reduce modulo p to the Dyer-Lashof operations.

3.4. THEOREM. *There is a mapping of polynomial algebras* $P_* = P^0_* + P^1_* + \cdots : H_*(BU, \mathbf{Z}_{(p)}) \to H_*(BU, \mathbf{Z}_{(p)})$ *defined inductively for n prime to p and* $k \geq 0$ *by*

$$\sum_{s=0}^{\infty} \binom{np^k - s(p-1)}{s} \frac{np^k}{np^k - s(p-1)} d_{e_n p^k - s(p-1)} = T_k(P_* a_{n,0}, \ldots, P_* a_{n,k}).$$

The component maps $P^s_*\colon H_m(BU, \mathbf{Z}_{(p)}) \to H_{m-2s(p-1)}(BU, \mathbf{Z}_{(p)})$ *satisfy multiplicative and comultiplicative Cartan formulae and reduce* mod p *to the dual Steenrod operations. In fact,* $P^s = \mathrm{Hom}(P^s_*, 1)$ *under the identification* $H^*(BU, \mathbf{Z}_{(p)}) = \mathrm{Hom}_{\mathbf{Z}_{(p)}}(H_*(BU, \mathbf{Z}_{(p)}), \mathbf{Z}_{(p)})$.

PROOF. P_* is shown to be well defined exactly as in the proof of Theorem 3.3. By

Corollary 3.2 and Theorem 3.3 the dual of the $\mathbf{Z}_{(p)}$ lift of P constructed in Theorem 3.3 is a map of polynomial Hopf algebras which agrees with P_* on the primitives and hence equals it. □

4. Dyer-Lashof operations on BU. For any infinite loop space X let $Q^r: H_m(X, \mathbf{Z}/p) \to H_{m+2r(p-1)}(X, \mathbf{Z}/p)$ denote the Dyer-Lashof operation. In constructing a lift to $\mathbf{Z}_{(p)}$ homology when $X = BU$ we use as our starting point the following algorithm of Kochman [8, Theorem 97].

4.1. THEOREM. *There is an inductive algorithm for computing $Q^r(d_n)$ using the following properties of Dyer-Lashof operations on $H^*(BU, \mathbf{Z}/p)$:*

(a) $Q^r: H_m(BU, \mathbf{Z}/p) \to H_{m+2r(p-1)}(BU, \mathbf{Z}/p)$ *is linear.*

(b) $Q^r d_m = 0$ *if* $m > r$.

(c) $Q^r(xy) = \sum_i Q^i x\, Q^{r-i} y$ *(Cartan formula).*

(d) $\Delta_* Q^r x = \sum_i \sum Q^i x' \otimes Q^{r-i} x''$ *where* Δ *is the diagonal and* $\Delta_* x = \sum x' \otimes x''$ *(co-Cartan formula).*

(e) $Q^r d_r = d_r^p$ *for all* $r > 0$.

(f) $P_*^s Q^r = \sum_i (-1)^i \binom{(r-s)(p-1)}{s-pi} Q^{r-s+i} P_*^i$ *(Nishida relation).*

(g) $Q^r d_{e_m} = (-1)^{r+m}\binom{r-1}{m-1} d_{e_{m+r(p-1)}}$.

(h) $Q^r d_m = (-1)^{r+m+1}\binom{r-1}{m} d_{m+r(p-1)}$ + *decomposables.*

The Dyer-Lashof operations also satisfy certain naturality conditions and an Adem type relation. We will use condition (g) and the Witt polynomial to construct a lift of Q^r just as we did for Steenrod operations. Identifying the mod p reduction is a lot harder this time, though, since we no longer have a Borel-Serre theorem giving $Q^r d_n$ as the mod p reduction of an integral class with nice coproduct.

Our construction uses, as before, formal sums $Q^0 + Q^1 + \cdots$. But by (g) or (h) $Q^r x$ will in general be nonzero for infinitely many r, so we work in the ring $H_{**}(BU, \mathbf{Z}_{(p)})$ of formal series $x_0 + x_1 + \cdots$ where $x_i \in H_{2i}(BU, \mathbf{Z}_{(p)})$. Finally, recall again our convention that the binomial coefficient $\binom{a}{b}$ vanishes if $b < 0$ or $a < b$.

4.2. THEOREM. *Let* $Q = Q^0 + Q^1 + \cdots : H_*(BU, \mathbf{Z}_{(p)}) \to H_{**}(BU, \mathbf{Z}_{(p)})$ *be the ring homomorphism defined inductively for n prime to p and $k \geq 0$ by*

$$\sum_{r=0}^{\infty} (-1)^{n+r}\binom{r-1}{np^k-1} d_{e_n p^k + r(p-1)} = T_k(Qa_{n,0}, \ldots, Qa_{n,k}).$$

The component maps $Q^r: H_m(BU, \mathbf{Z}_{(p)}) \to H_{m+2r(p-1)}(BU, \mathbf{Z}_{(p)})$ satisfy Cartan and co-Cartan formulae, vanish if $m > r$, and reduce modulo p to the Dyer-Lashof maps.

PROOF. The fact that Q is well defined follows just as in Theorem 3.3 using the following ((2) follows from Proposition 5.2):

$$r\binom{r-1}{np^k-1} = \binom{r}{np^k} \cdot np^k \equiv 0 \mod p^k. \tag{1}$$

$$\binom{rp-1}{np^k-1} - \binom{r-1}{np^{k-1}-1} = \frac{np^k}{rp}\left(\binom{rp}{np^k} - \binom{r}{np^{k-1}}\right) \equiv 0 \mod p^k. \tag{2}$$

We check that the mod p reductions of the component maps Q^r satisfy the conditions of Theorem 4.1 and hence equal the Dyer-Lashof operations.

By definition we have forced (g), and since $d_{e_n} = (-1)^{n+1}nd_n$ + decomposables by Lemma 2.2 it follows that

$$Q^r d_n = (-1)^{n+r}\binom{r-1}{n-1}\frac{n+r(p-1)}{n}\, d_{n+r(p-1)} + \text{decomposables.}$$

But

$$(-1)^{n+r}\binom{r-1}{n-1}\frac{n+r(p-1)}{n} - (-1)^{n+r+1}\binom{r-1}{n} = (-1)^{r+n}\binom{r}{n}\cdot p$$

and (h) is satisfied. The maps Q^r satisfy the Cartan formula in $H_*(BU, \mathbf{Z}_{(p)})$ since Q is multiplicative, and they satisfy the co-Cartan formula since $H_*(BU, Q)$ is primitively generated, Q is multiplicative and sends primitives to primitives. Since $Q^r d_{e_m} = 0$ for $m > r$ by definition, a straightforward application of Lemma 2.2 and the Cartan formula shows that $Q^r d_m = 0$ if $m > r$.

To prove (e), note first that $Q^1 d_1 = Q^1 d_{e_1} = d_{e_p}$, and

$$d_{e_p} = \sum_{w(\alpha)=p} (-1)^{p+|\alpha|}\frac{p}{|\alpha|}\{\alpha\}\, d^\alpha \equiv d^{pe_1} = d_1^p.$$

Assume inductively that $Q^i d_i \equiv d_i^p \bmod p$ for all $i < r$, and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be an r-tuple of weight r. Then

$$\begin{aligned} Q^r d^\alpha &= \sum_{i_1+\cdots+i_r=r} Q^{i_1}(d_1^{\alpha_1}) \cdots Q^{i_r}(d_r^{\alpha_r}) \\ &= Q^{\alpha_1}(d_1^{\alpha_1})Q^{2\alpha_2}(d_2^{\alpha_2}) \cdots Q^{r\alpha_r}(d_r^{\alpha_r}). \end{aligned}$$

The second equality follows from the fact that for any other partition we must have $i_j < j\alpha_j$ for some j and hence $Q^{i_j}(d_j^{\alpha_j}) = 0$ by the Cartan formula. If $\alpha \neq e_r$, then by the Cartan formula again, property (b) and induction it follows that $Q^{j\alpha_j}(d_j^{\alpha_j}) = d_j^{p\alpha_j}$ for each $j = 1, \ldots, r-1$. Thus

$$\begin{aligned} Q^r((-1)^{r+1}rd_r) &= Q^r(d_{e_r} - \sum_{\substack{\alpha\neq e_r \\ w(\alpha)=r}} (-1)^{r+|\alpha|}\frac{r}{|\alpha|}\{\alpha\}\, d^\alpha) \\ &= d_{e_r p} - \sum_{\substack{\alpha\neq e_r \\ w(\alpha)=r}} (-1)^{r+|\alpha|}\frac{r}{|\alpha|}\{\alpha\}\, d^{p\alpha} \\ &= \sum_{w(\alpha)=pr} (-1)^{r+|\alpha|}\frac{pr}{|\alpha|}\{\alpha\}\, d^\alpha - \sum_{\substack{\alpha\neq e_r \\ w(\alpha)=r}} (-1)^{r+|\alpha|}\frac{r}{|\alpha|}\{\alpha\}\, d^{p\alpha}. \end{aligned}$$

If $\alpha \neq p\beta$ for any r-tuple β then $\{\alpha\}/|\alpha| \in \mathbf{Z}_{(p)}$ by Proposition 5.1 so that $(pr/|\alpha|)\{\alpha\}$ d^α vanishes mod p. Thus the above difference of sums is congruent mod p to

$$(-1)^{r+p}rd_r^p + \sum_{\substack{w(\alpha)=r \\ \alpha\neq e_r}} (-1)^{r+|\alpha|}\left(\frac{pr}{|p\alpha|}\{p\alpha\} - \frac{r}{|\alpha|}\{\alpha\}\right) d^{p\alpha}$$

which equals $(-1)^{r+1}rd_r^p \bmod p$ by Proposition 5.2, proving (e).

We are left with verifying the Nishida relations. Let $\Phi_1^{r,s}$, $\Phi_2^{r,s}$: $H_*(BU, \mathbf{Z}_{(p)}) \to H_*(BU, \mathbf{Z}_{(p)})$ be defined by

$$\Phi_1^{r,s} = P_*^s Q^r \quad \text{and}$$
$$\Phi_2^{r,s} = \sum_i (-1)^{i+s} \binom{(r-s)(p-1)}{s-pi} Q^{r-s+i} P_*^i.$$

We show that $\Phi^{r,s} = \Phi_1^{r,s} - \Phi_2^{r,s}$ vanishes identically mod p.

ASSERTION 1 (DOUBLE CARTAN FORMULA).

$$\Phi_i^{r,s}(xy) = \sum_{k,l} (\Phi_i^{k,l}x)(\Phi_i^{r-k,s-l}y).$$

PROOF. Since the integral lifts P_*^s and Q^r satisfy a Cartan formula, $\Phi_1^{r,s}$ clearly satisfies a double Cartan formula. For $\Phi_2^{r,s}$ note that

$$\begin{aligned}
\Phi_2^{r,s}(xy) &= \sum_i (-1)^{i+s} \binom{(r-s)(p-1)}{s-pi} Q^{r-s+i} P_*^i(xy) \\
&= \sum_{i+j} (-1)^{i+j+s} \binom{(r-s)(p-1)}{s-p(i+j)} Q^{r-s+i+j} \left(\sum_j (P_*^i x)(P_*^j y) \right) \\
&= \sum_{i,j,k} (-1)^{i+j+s} \binom{(r-s)(p-1)}{s-p(i+j)} (Q^{k+i} P_*^i x)(Q^{r-s-k+j} P_*^j y) \\
&= \sum_{i,j,(k-l)} (-1)^{i+j+s} \left(\sum_l \binom{(k-l)(p-1)}{l-pi} \binom{(r-s-k+l)(p-1)}{s-l-pj} \right) \\
&\qquad \cdot (Q^{k-l+i} P_*^i x)(Q^{r-s-k+l+j} P_*^j y) \\
&= \sum_{k,l} (\Phi_2^{k,l}x)(\Phi_2^{r-k,s-l}y).
\end{aligned}$$

This uses the identity $\binom{a+b}{c} = \sum \binom{a}{l}\binom{b}{c-l}$ which follows by applying the binomial theorem to both sides of $(x+y)^{a+b} = (x+y)^a(x+y)^b$.

ASSERTION 2. If $\Phi^{r,s}x \equiv 0 \bmod p$ for all r, s, then $\Phi^{r,s}(x^{p^j}) \equiv 0 \bmod p^{j+1}$.

PROOF. From Assertion 1 it follows that $\Phi_i^r = \sum_s \Phi_i^{r,s}$ satisfies a Cartan formula, $i = 1, 2$. Suppose $\deg x > 0$ ($\deg x = 0$ case is clear) so that $\Phi_i^0 x = 0$. Then

$$\Phi_i^r(x^{p^j}) = \sum_{\substack{w(\alpha)=r \\ |\alpha|=p^j}} \{\alpha\}(\Phi_i x)^\alpha$$

where $(\Phi_i x)^\alpha = (\Phi_i^1 x)^{\alpha_1} \cdots (\Phi_i^r x)^{\alpha_r}$. By assumption there is a formal series y_r such that $\Phi_1^r x = \Phi_2^r x + p y_r$. Suppose, for some α, that p^k exactly divides α. Then p^{j-k} divides $\{\alpha\}$ by Proposition 5.1. It follows that

$$\{\alpha\}(\Phi_1^i x)^{\alpha_i} = \{\alpha\}(\Phi_2^i x)^{\alpha_i} + \sum_{l=1}^{\alpha_i} \{\alpha\}\binom{\alpha_i}{l} p^l y_r^l (\Phi_2^i)^{\alpha_i - l}$$

where each coefficient in $\sum_{l=1}^{\alpha_i}$ vanishes at least mod p^{j+1} by Proposition 5.1. Thus

$$\Phi_1^r(x^{p^j}) \equiv \sum_{\substack{w(\alpha)=r \\ |\alpha|=p^j}} \{\alpha\}(\Phi_2 x)^\alpha = \Phi_2^r(x^{p^j}) \mod p^{j+1}.$$

The assertion follows by taking components.

ASSERTION 3. $\Phi^{r,s} d_{e_n p^k} \equiv 0 \bmod p^{k+1}$.

PROOF. If $r = 0$ then $\Phi_1^{r,s} d_{e_m} = 0 = \Phi_2^{r,s} d_{e_m}$, while if $s = 0$ we have $\Phi_1^{r,s} d_{e_m} = Q^r d_{e_m} = \Phi_2^{r,s} d_{e_m}$. In general, since

$$Q^r d_{e_m} = (-1)^{m+r}\binom{r-1}{m-1} d_{e_{m+r(p-1)}}$$

and

$$P_*^s d_{e_m} = \binom{m-s(p-1)}{s} \frac{m}{m-s(p-1)}\, d_{e_{m-s(p-1)}},$$

the proof of the assertion reduces to showing that

$$\binom{r-1}{m-1}\binom{m+(r-s)(p-1)}{s} \equiv \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{m-i(p-1)}{i}\binom{r-s+i-1}{m-i(p-1)-1} \frac{m}{m-i(p-1)}$$

modulo p^{k+1} if p^k divides m and $r, s > 0$. This is established in the next section (Proposition 5.4).

To verify the Nishida relations it suffices by Assertion 1 to show that $\Phi^{r,s} a_{n,k} \equiv 0$ mod p for all $r, s \geq 0$, n prime to p, and $k \geq 0$. When $k = 0$ this is just Assertion 3 with $k = 0$, so we assume inductively that $\Phi^{r,s} a_{n,j} \equiv 0$ mod p for $0 \leq j < k$. Then

$$p^k \Phi^{r,s}(a_{n,k}) = \Phi^{r,s}(d_{e_n p^k}) - \Phi^{r,s}(a_{n,0}^{p^k}) - \cdots - p^{k-1}\Phi^{r,s}(a_{n,k-1}^{p}).$$

But each term on the right vanishes mod p^{k+1}, the first by Assertion 3 and the remaining terms by Assertion 2 and induction. Thus $\Phi^{r,s} a_{n,k} \equiv 0$ mod p, as desired. □

5. Appendix—some p local properties of multinomial coefficients. For any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\{\alpha\} = |\alpha|!/(\alpha_1! \cdots \alpha_n!)$. We say that p^k divides α if it divides each α_i, and it exactly divides α if no higher power divides α.

5.1. PROPOSITION. *If p^k divides $|\alpha|$ and p^j exactly divides α, $j \leq k$, then p^{k-j} divides $\{\alpha\}$.*

PROOF. Since $\{\alpha\} = \{\alpha_1, \alpha_2 + \cdots + \alpha_n\}\{\alpha_2, \ldots, \alpha_n\}$ it suffices to verify the case $n = 2$. If $\alpha = (\alpha_1, \alpha_2)$ with α_1 prime to p the result is well known, so suppose $\alpha = p^j\beta$ with β_1 prime to p. Let $\prod(i)$ denote the product of all natural numbers $< i$ which are prime to p. By some simple bookkeeping we obtain

$$\{p\beta\}/\{\beta\} = \prod(p\beta_1 + p\beta_2)/\prod(p\beta_1)\prod(p\beta_2)$$

and hence

$$\{p^j\beta\} \prod_{i=1}^{j} \prod(p^i\beta_1)\prod(p^i\beta_2) = \{\beta\} \prod_{i=1}^{j} \prod(p^i(\beta_1+\beta_2)).$$

Since $\prod$ takes values prime to p, the p divisibility of $\{p^j\beta\}$ equals that of $\{\beta\}$ and hence is at least p^{k-j}. □

5.2. PROPOSITION. *If p^k divides $|\alpha|$, then $\{p\alpha\} \equiv \{\alpha\}$ mod p^{k+1}.*

PROOF. Generalizing the formula in the proof above, for any n-tuple α we have $\{p\alpha\}/\{\alpha\} = \prod(|p\alpha|)/\prod(p\alpha_1)\cdots\prod(p\alpha_n)$ and hence $\prod(p\alpha_1)\cdots\prod(p\alpha_n)(\{p\alpha\}-\{\alpha\}) = \{\alpha\}(\prod(|p\alpha|) - \prod(p\alpha_1)\cdots\prod(p\alpha_n))$. Suppose p^j exactly divides α, so that p^{k-j} divides $\{\alpha\}$ by Proposition 5.1. But by Wilson's Theorem [6] we have that $\prod(|p\alpha|) \equiv (-1)^{|\alpha|/p^j} \bmod p^{j+1}$ and $\prod(p\alpha_i) \equiv (-1)^{\alpha_i/p^j} \bmod p^{j+1}$. Since $\prod$ takes values prime to p, p^{j+1} divides $\{p\alpha\}-\{\alpha\}$. □

The proof of the following result, which is essential in establishing the Nishida relations on the primitives, was supplied by L. Carlitz.

5.3. PROPOSITION. *Let a, b, c, d be nonnegative integers, $a \neq 0$. Then*

$$\sum_j \binom{b}{a-j}\binom{d}{c-j}\binom{b+d+j-1}{j} = \binom{a+d}{c}\binom{b+c-1}{a-1}\frac{bd+ab+cd}{a(a+d)}.$$

PROOF. For any m and integer $j > 0$ define $(m)_j = m(m+1)\cdots(m+j-1)$, and set $(m)_0 = 1$. Then $\binom{m}{k} = (-m)_k/(-k)_k$, and using $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$ it follows that

$$\begin{aligned}
S &= \sum_j \binom{b}{a-j}\binom{d}{c-j}\binom{b+d+j-1}{j} \\
&= \sum_j \binom{b}{a-j}\binom{d}{c-j}\binom{b+d+j}{j} - \sum_j \binom{b}{a-j-1}\binom{d}{c-j-1}\binom{b+d+j}{j} \\
&= \binom{b}{a}\binom{d}{c}\sum_j \frac{(-a)_j(-c)_j(b+d+1)_j}{j!(b-a+1)_j(d-c+1)_j} \\
&\quad - \binom{b}{a-1}\binom{d}{c-1}\sum_j \frac{(-a+1)_j(-c+1)_j(b+d+1)_j}{j!(b-a+2)_j(d-c+2)_j}.
\end{aligned}$$

To get a closed form for the above sums we appeal to the theorem of Saalschütz [19, p. 87]: If q, r, s, t are integers with $q \geq 0$ then

$$\sum_j \frac{(-q)_j(r)_j(s)_j}{j!(t)_j(r+s-q-t+1)_j} = \frac{(t-r)_q(t-s)_q}{(t)_q(t-r-s)_q}.$$

Applying this to each of the sums above yields

$$\begin{aligned}
S &= \binom{b}{a}\binom{d}{c}\frac{(d+1)_a(-b-c)_a}{(d-c+1)_a(-b)_a} - \binom{b}{a-1}\binom{d}{c-1}\frac{(d+1)_{a-1}(-b-c+1)_{a-1}}{(d-c+2)_{a-1}(-b)_{a-1}} \\
&= \frac{b!}{a!(b-a)!}\,\frac{d!}{c!(d-c)!}\,\frac{(a+d)!(d-c)!}{d!(d-c+a)!}\,\frac{(b-a)!(b+c)!}{b!(b+c-a)!} \\
&\quad - \frac{b!}{(a-1)!(b-a+1)!}\,\frac{d!}{(c-1)!(d-c+1)!} \\
&\quad \cdot \frac{(a+d-1)!(d-c+1)!}{d!(d-c+a)!}\,\frac{(b-a+1)!(b+c-1)!}{b!(b+c-a)!} \\
&= \binom{a+d}{c}\binom{b+c-1}{a-1}\frac{bd+ab+cd}{a(a+d)}. \quad \square
\end{aligned}$$

The binomial coefficient $\binom{m}{k}$ is often defined for any complex number m and integer $k \geq 0$ by $\binom{m}{k} = (-m)_k/(-k)_k$. With this definition the above result and proof are valid for any nonzero complex numbers a, d with $a + d \neq 0$ and integers c, d.

We have not adopted that convention here since, for example, the Nishida relations would then be incorrect if $s > r$.

5.4. PROPOSITION. *Suppose m, r, s are positive integers such that $m + (r - s)(p - 1) > 0$ and p^k divides m. Then*

$$\binom{r-1}{m-1}\binom{m+(r-s)(p-1)}{s}\frac{m+r(p-1)}{m+(r-s)(p-1)}$$
$$\equiv \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{m-i(p-1)}{i}\binom{r-s+i-1}{m-i(p-1)-1}\frac{m}{m-i(p-1)} \mod p^{k+1}.$$

The above sum is assumed taken over all values of i such that the binomial coefficients involved are nonzero. In particular, $m - i(p - 1) \geq 1$ for all such i.

PROOF. By our conventions on binomial coefficients it is easy to check that if $s > r$ the above congruence is actually $0 = 0$. When $s = r$ it reduces to the identity $p = p$ or $0 = 0$ depending on whether or not $m = r = s \equiv 0 \bmod p$. Thus suppose $r > s$. Using the identity $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{b-a}{c-a}$ it follows that

$$R = \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{m-i(p-1)}{i}\binom{r-s+i-1}{m-i(p-1)-1}\frac{m}{m-i(p-1)}$$
$$= \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{m-i(p-1)}{i}\binom{r-s+i}{m-i(p-1)}\frac{m}{r-s+i}$$
$$= \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{r-s+i}{i}\binom{r-s}{m-ip}\frac{m}{r-s+i}.$$

But note that if p^l divides $r - s + i$ then

$$\binom{r-s+i}{i} \equiv \binom{pr-ps+pi}{pi} \mod p^{l+1}$$

by Proposition 5.2 and hence

$$\binom{r-s+i}{i}\frac{m}{r-s+i} \equiv \binom{p(r-s)+pi}{pi}\frac{pm}{p(r-s)+pi} \mod p^{k+1}.$$

Thus

$$R \equiv \sum_i \binom{(r-s)(p-1)}{s-pi}\binom{(r-s)p+pi}{pi}\binom{r-s}{m-ip}\frac{pm}{p(r-s+i)} \mod p^{k+1}$$
$$\equiv \sum_j \binom{(r-s)(p-1)}{s-j}\binom{(r-s)p+j}{j}\binom{r-s}{m-j}\frac{pm}{p(r-s)+j} \mod p^{k+1}$$
$$= \sum_j \binom{(r-s)(p-1)}{s-j}\binom{(r-s)p+j-1}{j}\binom{r-s}{m-j}\frac{m}{r-s}$$

where the second line is congruent to the first modulo p^{k+1} since we have added terms with j prime to p so that $mp/[(r - s)p + j] \equiv 0 \bmod p^{k+1}$. The desired result now follows from Proposition 5.3 by setting $a = m, b = (r - s)$, $c = s$, and $d = (r - s) \cdot (p - 1)$. □

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Department of Mathematics, State University of New York at Albany, Albany, New York 12222